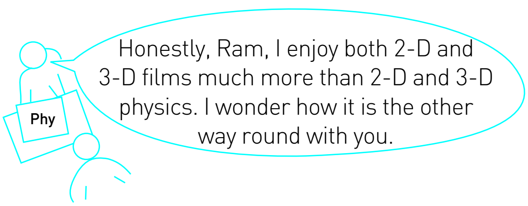
MOTION IN TWO AND THREE DIMENSIONS

PC Agarwal

Associate Professor of Physics
Regional Institute of Education,
Ajmer

Introduction

We live and work in a three-dimensional world. Laws of physics enable us to describe motions of bodies in one, two and three dimensions. Though motions in two and three dimensions may appear to be extensions of motion in one dimension, there are subtle differences between them, and special kinds of motion, like rotation, are possible in two and three dimensions only. Moreover vectors can be used to describe motions in two and three dimensions. As such students have difficulties even in one-dimensional motion. Because of the additional complications, motions in two and three dimensions become much harder for the learners. Lack of adequate thinking and visualisation leads to misconceptions or alternative conceptions not only in the minds of students but also in the minds of the teachers.



In this section, we would like to discuss some important aspects of motion in two and three dimensions vis-à-vis that in one dimension. The focus shall be on the conceptual hard spots, which are often ignored in textbooks. We shall address here only some of the ideas and issues relevant to the higher secondary level, such as *vectors, reference frames, coordinate systems* in the current article as a part under the title motions in two and three dimensions, while discussion on other issues will be given in the coming articles. In the discussion of motion along a straight line, the displacement of a point from a position x_1 to x_2 on the line joining these two positions is independent of the origin chosen on the line. Because of this fact, such a displacement can be considered as an entity on its own, without reference at all to the origin. As such, it is specified completely by a *length* and a *sense,* since displacements are considered positively conventionally when motion is from left to right, and negative when motion is from right to left. Thus an *arrow* having a length equal to the magnitude of the displacement and having arrowhead in the direction of motion can be used to represent the displacement. For any choice of origin, both length and sense are given simply by $x_2 - x_1$; changing the origin (i.e., the reference point about which studies are being made) will simply add the same constant quantity to both x_1 and x_2 and without in any way changing the displacement. For this reason, the usual notation of elementary algebra is quite sufficient for the representation of displacements along a line. But when motion is generalised to that in two or three dimensions, introduction of a new notation, vector, is convenient.

1. Vectors

For a displacement represented by an arrow, as in Fig.1, the notation $\vec{x}$ is standard although it is often convenient to call it simply $\overrightarrow{TH}$ when referring to a diagram. *T* is the 'tail' and *H* is the 'head' of the arrow.

Fig. 1

With this convention, the displacement $-\vec{x}$ is just $\overrightarrow{HT}$. Because the origin has been dispensed with, no meaning can be attached to the 'position' of the displacement; it may be drawn anywhere along the line. A quantity considered in this way is called a *vector* and can be represented by an arrow or a line with an arrowhead and of a definite length. Thus displacement is a vector. Quantities that do not require a direction or *sense* are called *scalars*. Thus ordinary numbers are scalars; distance and speed are scalars. Time is also considered as a scalar quantity. Since velocity is displacement divided by a time and acceleration is velocity divided by a time, both velocity and acceleration are vectors and may be represented by arrows and symbols $\vec{v}$ and $\vec{a}$ respectively.

Time appears to have characteristics somewhat similar to those of the displacement vector $\vec{x}$. In fact, time (t) can be plotted along a time axis or t-axis just like the coordinate *x* along the *x*-axis. One may choose the reference point or origin ($t = 0$) at any instant or at any point on the t-axis, and from that instant to future is time forwards (t *+ve*) whereas from that instant to past is time backwards (t *-ve*). Thus time appears to have a 'direction' as well as a magnitude like $\vec{x}$. So, some may argue, why do we consider time as a scalar and not as a vector? The reason is that a vector like $\vec{x}$, $\vec{v}$, or $\vec{a}$ has a direction in space, say *north, south,* 30^0 *to east,* etc. But time has no such direction in space. A point in space does not have a time coordinate. Moreover, whereas a vector has as many components as there are dimensions defining the space (1, 2 or 3 as the case may be), time has no components; it is denoted by a single notation *t*. A clock placed at point *P* in space keeps the same time as the one placed at another point *Q*, hence time is considered a scalar parameter, independent of the space coordinates in Newtonian mechanics.

Although a vector is considered as independent of the choice of the origin, it depends on the 'frame of reference' (FOR) from which one happens to regard it. This important idea of FOR may be explained as follows. The displacement shown in Fig.1 may be thought of as the displacement of a fly that has crawled horizontally across the page of the book. The page of the book is the FOR for this displacement and this has no consequence where the origin is located on the line of the arrow. But, as the fly crawls on the page, the whole book may be moved from left to right on the table. The table is a different FOR, and the displacement of the fly seen from the table will not agree with that seen from the page of the book. If the displacement of the book on the table is denoted by $\vec{X}$, then, with reference to the table, i.e., as seen from the table, the displacement of the fly will be $\vec{X} + \vec{x}$, where $\vec{x}$ denotes the displacement of the fly on the page of the book.

Similarly, we can treat the velocity vector. If the velocity of the fly relative to the page is $\vec{v}$ and the velocity of the book relative to the table is $\vec{v}$, then the velocity of the fly relative to the table will be $\vec{v} + \vec{V}$.

So long as the motion is along a straight line, all of the above reasoning is perfectly obvious and already contained in the concept of *x* as a coordinate. Nevertheless, the new notation introduced is a very powerful aid to think when we tackle problems in two and three dimensions. There are many more directions in a plane or in space than just the two possibilities in the case of one-dimensional motion. Therefore, a sound knowledge of vector notation and method is regarded as essential for all science students and teachers. This can be learnt easily with the help of knowledge of reference frames (or FOR) and coordinate systems.

Mathematically there is no difference between one-dimensional vectors and scalars which admit +ve, -ve and zero values. Rules of operation are similar in both the cases

Alternative conceptions/weak points and possible remedies:

- Relative sizes and orientations of vectors are often incorrectly represented in diagrams by students.

  For example, when two vectors of unequal magnitudes are given, the one with greater magnitude should not look similar to or shorter than the other in their diagrammatic representation. For example, when angle between two vectors is given to be 30^0, the same should not look like 60^0 or 90^0 in the diagram. For that matter, a straight line should look like a straight line to the extent possible. Unmindful-ness or carelessness might be the reason for the situation. Though exact accuracy is not possible, nor essential, care must be exercised while drawing vector diagrams lest incorrect diagrams should induce incorrect thinking. The teacher should not only impress upon the learners the importance of this aspect but should also make use of it while teaching vectors.

- Pupils often ignore or forget to attach the vector symbol (arrowhead) to a vectorial quantity. They ought to recognise that though it is possible to represent vectors by **boldface** letters in print form, as is done in many books on physics, the arrowhead option is easier and more practicable while writing by hand.

- Another error commonly committed is that in a vectorial equation, the left hand side correctly denotes a vector whereas the right hand side is devoid of the corresponding symbols or notations, and vice versa. The teacher as well as the students ought to note that this is inconsistent and incorrect both mathematically and physically, and hence should be avoided.

2. Frame of Reference (FOR)

We accept the local surroundings – a collection of objects attached to the earth and therefore at rest relative to one another – as defining a FOR. This is a reference frame with respect to which changes of position of other objects can be observed and measured. It is clear that the choice of FOR, to which the motion of an object is referred, is entirely a matter of taste and convenience, but it is often advantageous to use a FOR in which description of motion is the simplest. A very commonly used FOR is the 'laboratory frame' or 'lab frame', i.e., a FOR attached to the laboratory in which motions of bodies are observed. Another useful FOR is the 'centre of mass frame' or the 'CM frame' attached to the centre of mass of the system. There are powerful theoretical reasons for preferring some FOR to others. The 'best' choice of FOR becomes ultimately a question of dynamics i.e., dependence on actual laws of motion and force. But the choice of a particular FOR is often made without regard to the dynamics, and for the present, we shall just concern ourselves with the purely kinematics problems of analysing positions and motions with respect to any given frame.

Alternative conceptions / weak points and possible remedies:

In 1994, a Bombay (now Mumbai) group of researchers (Panse et. al., 1994) studied alternative conceptions of students in Galilean relativity in the context of FOR. Though their subjects were physics undergraduates, some of their findings are relevant for higher secondary level too, both in Newtonian or non-relativistic as well as relativistic mechanics.

- Students in general have a hazy idea about the concepts of FOR and observer. The following simple activity may help them in this regard.

Activity

Draw a circle on a piece of paper placed flat on a table. Look at it vertically down. Clearly you see the circle. Now, if you tilt your head at different angles with respect to the paper, the circle may look like ellipses of different eccentricities. For some position of your head, the circle may even look like a straight line! Remember that it is the same circle being looked at from different angles or with different perspectives. Each position of your head may be considered to be a different FOR. Although you are the one who observes all this, each such FOR can be associated with an independent 'observer'. Clearly, the same thing may look different to observers in different FOR.

- Students have a tendency to think that an observer has to be a human being (anthropomorphic view). This is in spite of the statement made in standard textbooks that an observer can be either a person or an instrument.

It is important to recognise that measurements, which comprise observations, are carried out by impersonal instruments, and the persons behind the instruments are irrelevant for physics (unless you want to include the effects of human errors in the measurements). This reflects the principle of *objectivity in science*. It may thus be better to replace the notion of 'observer' by the notion of FOR. For example, 'a passenger in a train' may be replaced by the 'train's FOR' and 'observer on ground' by 'ground's FOR' etc.

- Students often picture a FOR as something *physically* attached to the observer or on object and *localised* by the physical extensions of the object.

This misconception should be removed as early as possible. FOR is an *abstract concept* and is not something *attached* to the observer or on object, nor is it *limited* by the physical extensions of the object. It should be *imagined* or *visualised.* For instance, in a three-dimensional space, it is like a system of rigid rods sticking out from a point in three mutually perpendicular directions and stretching from - ∞ to + ∞. The origin of an object's FOR may be inside or outside the object. The origin can be anywhere in space and orientation of the axes denoting the FOR can be chosen arbitrarily. It ought to be recognised that the FOR shares no other property of the object except its state of motion.

Questions

In order to check the comprehension of the learners, the following questions may be put to them. The teacher should first find the answers himself/herself, compare them with those provided by the learners, and then conduct discussions as necessary.

(i) When a food packet is dropped from a helicopter during flood relief work, the packet is no longer at rest with the FOR of helicopter. True/False

If your answer is 'True', what would you do to keep the packet in the helicopter's FOR?

(ii) Fig. 2(a) and Fig. 2(b) depict two FORs for a box at rest. Tick the correct option.

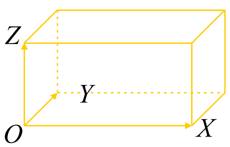


Fig. 2(a)

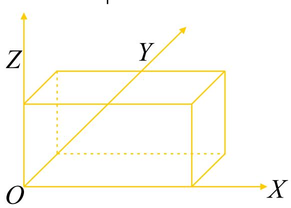


Fig. 2(b)

a) FOR of the box is given by Fig. 2 (a) only.

b) FOR of the box is given by Fig. 2 (b) only.

c) FOR of the box is given by both Fig. 2 (a) and 2 (b).

d) FOR of the box is given by none of these figures.

iii) Would you accept that frame as a FOR of the box, referred to above, whose origin lies outside the box? Why?

iv) While moving on your bicycle you raised your hand. This event occurred in the bicycle's FOR only. True / False,

Give reasons.

3. Coordinate Systems and Unit Vectors

A FOR, as we have said, is defined by some array of physical objects that remain at rest relative to one another. Within any such frame we make measurements of position and displacement by setting up a *coordinate system* of some kind. In doing this, we have a free choice of origin and of the kind of coordinate system that is best suited to the purpose at hand. Since the space of our experience has three dimensions, we must in general specify three separate quantities in order to fix uniquely the position of a point.

This will reduce to two separate quantities in a two-dimensional plane. The quantities are the Cartesian coordinates (x, y) or the polar coordinates (r,θ). The square grid basis of Cartesian coordinates is shown in Fig. 3(a) and the circular grid of the plane polar coordinates is shown in Fig. 3(b). These are the two standard coordinate systems used in two dimensions.

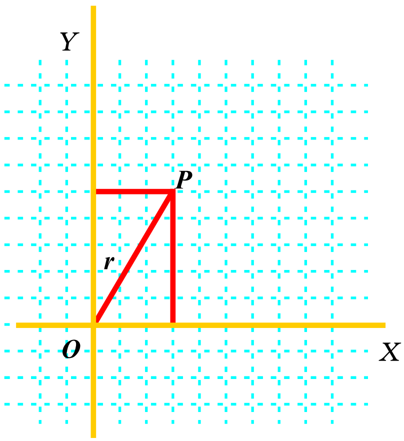


Fig. 3(a) Square Grid

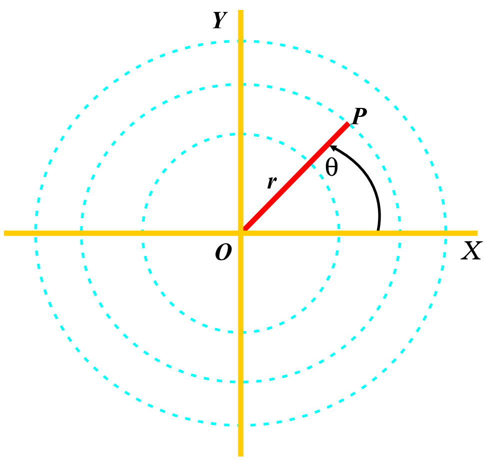


Fig. 3(b) Circular Grid

In polar coordinates, r varies form 0 to ∞ i.e. $0 < r < \infty$ and θ varies form 0 to 2π i.e. $0 < \theta < 2\pi$.

In two dimensions, with reference to Fig. 3 (a) and 3 (b), we have the familiar relations:

$r^2 = x^2 + y^2$, (1a)

$\tan\theta = y/x$, (1b)

$x = r\cos\theta,\ y = r\sin\theta$ (1c)

Similarly, in three dimensions, the standard systems used are Cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ) which are together presented in Fig. 4.

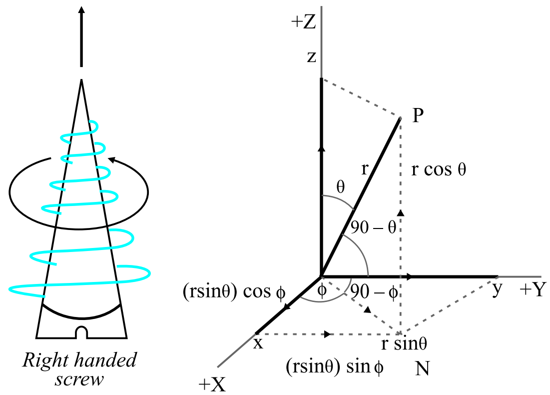


Fig. 4

Line *ON* is the projection of *r (OP)* on the *x-y* plane. The angles θ *and* ϕ are called polar angle and azimuth respectively. Similar to the set of equations in two dimensions [(1a) – (1c)], we have, in three dimensions,

$r^2 = x^2 + y^2 + z^2$, (2a)

$\tan\theta = \sqrt{(x^2 + y^2)}\,/z$ (2b)

$\tan\phi = y/x$, (2c)

$x = r\sin\theta\ \cos\phi,\ y = r\sin\theta\ \sin\phi,\ z = r\cos\theta.$ (2d)

The Cartesian system chosen here is a right-handed system by which we mean the following: the positive *Z* direction is chosen so that, looking upward along it, the process of rotating from the positive *X* direction toward the positive *Y* direction corresponds to that of a right-handed screw (see Fig. 4). It then follows that the cyclic permutations of this operation are also right-handed from *+Y to +Z looking along +X, and from +Z to +X looking* *along + Y.* You may note that the two-dimensional coordinate system, as shown in Fig. 3 (a) would, in this convention, be associated with a positive *Z* axis sticking up toward you out of the plane of the paper.

Now we come to the concept of *unit vectors.* Conventionally, the alphabets *A, B, C, ...* or *a, b, c, . . .* are used to represent constants, *U, V, W, X, Y, Z* OR *u, v, w, x, y, z* etc. to represent variables. Alphabets *a* to *z* are used to represent real numbers excluding *i, j, k, l, m, and n* which are used to represent integers. Usually, we make use of these integer symbols to represent unit vectors along the various coordinate axes or directions in the Cartesian systems. Various options are further exercised in this regard: some represent a unit vector in the form $\vec{i}$ (i with an arrowhead), some prefer the form **i** (bold *i*), whereas some others put it as $\hat{i}$ (*i* with a carat or cap or hat). Here we shall prefer the carat version since it is likely to produce minimal confusion and is easier to write, and shall denote unit vectors along the *X, Y* and *Z* axes as $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively. Then referring to Fig. 4 in three dimensions, we shall have the position vector of point *P* with respect to the origin *O* as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{3}$$

In two dimensions [Fig. 3 (a)], the position vector becomes

$$\vec{r} = x\hat{i} + y\hat{j} \tag{4}$$

In the plane polar coordinate system [Fig. 3 (b)], one uses the symbol e_r to denote a unit vector in the direction of increasing *r* at constant θ, and the symbol e_θ to denote a unit vector in the direction of increasing θ at right angles to $\vec{r}$. The use of the symbol *e* for this purpose comes from the German word 'Einheit' which stands for 'unit'. In this system of coordinates, the vector $\vec{r}$ is simply equal to re_r, and one might wonder why the unit vector e_θ is introduced at all. However, it becomes very important as soon as we consider motions rather than static displacements. For motions we will often have a component perpendicular to *r*. It may be noted that, even in Cartesian systems, some like to use the symbols e_x, e_y and e_z to denote unit vectors along the *X, Y* and *Z* axes respectively. Some even write these unit vectors as $\hat{x}$, $\hat{y}$ and $\hat{z}$ respectively. In the plane polar coordinate system, another version of unit vectors is possible: $\hat{r}$ and $\hat{\theta}$ in place of e_r and e_θ respectively. Obviously, the former is easier to write than the latter. We thus recognise that there are multiple acceptable ways of denoting unit vectors in a given coordinate system, and one can pick and choose depending upon convenience.

Alternative conceptions/weak points and possible remedies:

- Students often confuse between the various systems of unit vectors and the various symbols used for vectors and are likely to mix them up. The teacher should see that this does not happen. A good tactic is to instruct them to explain all symbols and notations, used by them, explicitly while preparing notes and answering questions. They should also practice to read and speak the notations correctly (for example, $\hat{i}$ as '*i* cap').
- The same symbol θ in two and three dimensions may be confusing. The students ought to recognise that this is a standard practice in mechanics and the two situations are distinct.
- Students often use the components ($r\cos\theta$) and ($r\sin\theta$) mechanically without correctly

locating the angle θ. This leads to wrong conclusions. Correct conceptualisation and visualisation of the given situation is necessary for avoiding similar problems and this can come through practice.

- It is not only difficult for the teacher to draw a three-dimensional coordinate system on the two-dimensional plane of a board, the students also find it difficult to visualise and perceive.

The following activity may help in this context.

Activity

Take a blob of plasticene or putty and place it on a plane like a table or a floor. By inserting straight sticks into it at appropriate angles, a three-dimensional space can be modelled. Various projections can also be depicted with the help of smaller size blobs and other sticks. Permanent hollow tube-like structures may also be prepared from a metal or plastic which can be used along with sticks or rods for the same purpose.

Questions

(i) In $x-y$ plane, a line of length *1 m* originates from the origin and makes an angle of 30^0 with the *y*-axis. What are the lengths of the corresponding *x* and *y* components?

(ii) In a three-dimensional space, a line of length *1 m* starts from the origin and makes an angle of 60^0 with respect to the $x-y$ plane. Its projection on the $x-y$ plane makes an angle of 30^0 with the *y*-axis. Find the lengths of the corresponding *x, y* and *z* components.

(iii) Is it possible to describe a two-dimensional plane by means of a grid of non-perpendicular straight lines? Give a demonstration.

(iv) Looking at the relations among the various variables in two and three dimensions, as quoted in our discussion above, how can you reduce a three-dimensional coordinate system to a two-dimensional one?

(v) Design a left-handed Cartesian coordinate system by looking at the right-handed Cartesian coordinate system we have discussed above.

(vi) Obtain expressions for e_r *and* e_θ (or equivalently $\hat{r}$ and $\hat{\theta}$) in terms of $\hat{i}$ *and* $\hat{j}$. Test that they are orthogonal to each other.

It is interesting to note that in Cartesian systems all the coordinates (*x, y, z*) have the dimensions of length, whereas in plane polar and spherical polar systems, only one of the coordinates (*r*) has the dimensions of length and others (like θ or θ and φ) are dimensionless angular measures. These are examples of what are called 'generalised coordinates' used very often in advanced mechanics.

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